Disorder points of the IRF and checkerboard Potts models

1984 J. Phys. A: Math. Gen. 17 L911

(http://iopscience.iop.org/0305-4470/17/17/001)

J. Phys. A: Math. Gen. 17 (1984) L911-L917. Printed in Great Britain

LETTER TO THE EDITOR

Disorder points of the IRF and checkerboard Potts models

R J Baxter†

Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

Received 7 August 1984

Abstract. Conditions are given for a general 'interactions-round-a-face' (IRF) model in statistical mechanics to have a disorder point, and expressions obtained for the free energy and intra-row correlations. These are applied to the checkerboard Potts model, thereby verifying a recent conjecture by Jaekel and Maillard.

A number of papers have appeared in the last 15 years that deal with disorder points of particular two-dimensional lattice models (Stephenson 1970, Welberry and Galbraith 1973, Welberry and Miller 1978, Verhagen 1976, Enting 1977 and 1978, Rujan 1982, Peschel and Rys 1982, Dhar 1982). Some use the statistical theory of Markov processes, others (in particular Rujan 1982) are based on the transfer matrix method of statistical mechanics.

Here I shall apply the latter technique to the general IRF model (Baxter 1980) on the square lattice (drawn diagonally). I shall define a disorder point as one at which the maximal eigenvector of the transfer matrix is a direct product of factors, each factor corresponding to an edge of the lattice. If this is true of both the right and left eigenvectors, then the correlations within a horizontal row are those of a one-dimensional nearest-neighbour model. These correlations must therefore decay exponentially with distance and there can be no long-range order, at least in the horizontal direction.

In this letter these methods are applied to the checkerboard Potts model (i.e. the Potts model on the square lattice, with alternating interactions). A recent conjecture by Jaekel and Maillard (1984) is thereby verified.

We now consider the IRF model. Draw the square lattice diagonally, as in figure 1. With each site i associate a 'spin' σ_i, which can take any prescribed set of values (e.g. or -1; or the integers $1, \ldots, q$; or 'red', 'white' and 'blue'). To each face assign a Boltzmann weight factor $w(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$, where i, j, k, l are the four sites round the face, arranged anti-clockwise as in figure 1. Then the partition function is

$$Z = \sum_{\{\sigma\}} \prod_{(i,j,k,l)} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l), \tag{1}$$

where the sum is over all values of all the spins, and the product is over all faces of the lattice.

Now consider a horizontal row of faces of the lattice, as in figure 2. Let $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_n\}$ be the set of lower spins, as indicated, and let $\boldsymbol{\sigma}' = \{\sigma_1', \ldots, \sigma_n'\}$ (with

† This work was performed while the author was visiting the Istituto per l'Interscambio Scientifico, Torino, Italy.

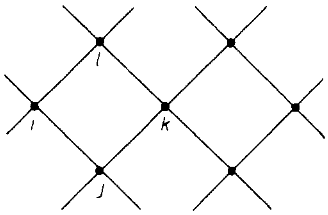

Figure 1. The square lattice, drawn diagonally, showing a typical face i, j, k, l.

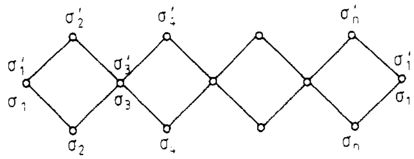

Figure 2. A horizontal row of faces of the lattice: spins $\sigma_1, \ldots, \sigma_n$ lie on sites in the centre and lower rows; $\sigma'_1, \ldots, \sigma'_n$ on sites in the centre and upper rows. If i is odd, $\sigma'_i = \sigma_i$.

$\sigma'_{2j-1} \equiv \sigma_{2j-1}$) be the upper spins. Then the product of the Boltzmann weights of the faces can be written as

$$V(\boldsymbol{\sigma}; \boldsymbol{\sigma}') = \prod_{j=1}^{n/2} w(\sigma_{2j-1}, \sigma_{2j}, \sigma_{2j+1}, \sigma'_{2j}) \delta(\sigma_{2j-1}, \sigma'_{2j-1}), \tag{2}$$

where we impose cyclic boundary conditions, so that $\sigma_{n+1} \equiv \sigma_1$; the integer n must be even.

We can regard the $V(\boldsymbol{\sigma}; \boldsymbol{\sigma}')$ as elements of a transfer matrix V that takes one from the upper spin set $\boldsymbol{\sigma}'$ to the lower set $\boldsymbol{\sigma}$. For the next row we have a similar transfer matrix W, except that now the positions $1, \ldots, n$ are displaced one unit to the right (or left). Thus

$$W(\sigma_1, \ldots, \sigma_n \,|\, \sigma'_1, \ldots, \sigma'_n) = V(\sigma_2, \ldots, \sigma_n, \sigma_1; \sigma'_2, \ldots, \sigma'_n, \sigma'_1). \tag{3}$$

In the usual way (Baxter 1980), we can now write the partition function for a lattice of $2m$ such rows (with mn sites) as

$$Z = \mathrm{Tr}(VW)^m. \tag{4}$$

For m large, it follows that

$$Z \sim \Lambda^{2m}, \tag{5}$$

where Λ^2 is the numerically largest eigenvalue of VW, and can be taken to be given by the pair of equations

$$Vy = \Lambda x, \qquad Wx = \Lambda y. \tag{6}$$

Here x, y are the right eigenvectors of VW and WV, respectively. For a physical model, all Boltzmann weights $w(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ must be non-negative, so from the Perron–Frobenius theorem (Frobenius 1908), it must be possible to normalise x and y so that all their elements are non-negative.

The first of equations (6) can be written more explicitly as

$$\sum_{\boldsymbol{\sigma}'} V(\boldsymbol{\sigma}; \boldsymbol{\sigma}') y(\sigma'_1, \ldots, \sigma'_n) = \Lambda x(\sigma_1, \ldots, \sigma_n), \tag{7}$$

where $x(\boldsymbol{\sigma})$, $y(\boldsymbol{\sigma})$ are the elements of x, y. In general we cannot solve equations (6), but we can look for the conditions under which $x(\boldsymbol{\sigma})$, $y(\boldsymbol{\sigma})$ have the simple product forms:

$$\begin{aligned} x(\sigma_1, \ldots, \sigma_n) &= f(\sigma_1, \sigma_2) g(\sigma_2, \sigma_3) f(\sigma_3, \sigma_4) \ldots g(\sigma_n, \sigma_1), \\ y(\sigma_1, \ldots, \sigma_n) &= g(\sigma_1, \sigma_2) f(\sigma_2, \sigma_3) g(\sigma_3, \sigma_4) \ldots f(\sigma_n, \sigma_1). \end{aligned} \tag{8}$$

Substituting these into (7), using (2), we find that the equation is satisfied if there exists a parameter κ and a single-spin function $\varphi(\sigma)$ such that

$$\Lambda = \kappa^{n/2} \tag{9}$$

and

$$\sum_{\sigma_2'} w(\sigma_1, \sigma_2, \sigma_3, \sigma_2')g(\sigma_1, \sigma_2')f(\sigma_2', \sigma_3) = \kappa\varphi(\sigma_1)f(\sigma_1, \sigma_2)g(\sigma_2, \sigma_3)/\varphi(\sigma_3) \tag{10}$$

for all values of σ_1, σ_2, σ_3.

Further, the second of the equations (6) is then also satisfied. From (5) and (6), we see that (for m large)

$$\kappa = Z^{1/mn}, \tag{11}$$

so κ is the partition function per site.

If each σ_i takes q values, then (10) is a set of q^3 equations. On the other hand, (10) is unchanged by renormalising $f(a, b)$, $g(a, b)$, $\varphi(a)$, or by multiplying them by $[u(a)v(b)]^{-1}$, $u(b)v(a)$, $u(a)v(a)$, respectively (for any functions $u(a)$, $v(a)$). It follows that we have only $2q^2 - q$ distinct unknowns in f, g, φ, κ at our disposal, so for a general Boltzmann weight function w we cannot satisfy (10).

However, for certain special functions w it is possible to satisfy (10). If $f(a, b)$ and $g(a, b)$ are non-negative, then Λ is the maximum eigenvalue of the transfer matrix, and κ is the partition function per site.

We can represent (10) pictorially as in figure 3: the LHS is the partition function of the square on the left, weights w, g, f being associated with the face and the top edges, their product being summed over the spin denoted by a full circle. The RHS of (10) is represented by the RHS of figure 3, weights φ, f, g, φ^{-1} being associated as shown with the sites and edges.

Correlations. So far we have discussed only the right eigenvectors x, y of VW and WV. Let $\tilde{x}$, $\tilde{y}$ be the left eigenvectors, i.e.

$$\tilde{x}V = \Lambda\tilde{y}, \qquad \tilde{y}W = \Lambda\tilde{x}. \tag{12}$$

They will be of the form (8) (with x, y, f, g) replaced by ($\tilde{x}$, $\tilde{y}$, $\tilde{f}$, $\tilde{g}$) if there exists $\tilde{\varphi}$ such that

$$\sum_{\sigma_2'} w(\sigma_1, \sigma_2', \sigma_3, \sigma_2)\tilde{f}(\sigma_1, \sigma_2')\tilde{g}(\sigma_2', \sigma_3) = \kappa\tilde{\varphi}(\sigma_1)\tilde{g}(\sigma_1, \sigma_2)\tilde{f}(\sigma_2, \sigma_3)/\tilde{\varphi}(\sigma_3). \tag{13}$$

For some models (in particular for the checkerboard Potts model), it turns out that if (10) can be satisfied, so can (13). We then know both the right and left eigenvectors of VW. The spins $\sigma_1, \ldots, \sigma_n$ in figure 2 all lie on a zig-zag horizontal line. Let $L(\sigma_1, \ldots, \sigma_n)$ be any function of these spins only. Then its expectation value is

$$\begin{aligned}\langle L\rangle &= \langle L(\sigma_1, \ldots, \sigma_n)\rangle \\ &= Z^{-1} \sum_{\{\sigma\}} L(\sigma_1, \ldots, \sigma_n) \prod_{(i,j,k,l)} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \\ &= Z^{-1} \sum_{\sigma_1,\ldots,\sigma_n} L(\sigma_1, \ldots, \sigma_n)[(VW)^m]_{\sigma_1,\ldots,\sigma_n|\sigma_n',\ldots,\sigma_n'} \\ &= Z^{-1}\,\mathrm{Tr} L(VW)^m,\end{aligned} \tag{14}$$

where L is the diagonal matrix with entries $L(\sigma_1, \ldots, \sigma_n)\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$. In the limit of m

large it follows that

$$\langle L \rangle = \tilde{x} L x / \tilde{x} x, \tag{15}$$

so from (8) and its analogue for $\tilde{x}$,

$$\langle L \rangle = S^{-1} \sum_{\sigma_1, \ldots, \sigma_n} L(\sigma_1, \ldots, \sigma_n) p(\sigma_1, \sigma_2) r(\sigma_2, \sigma_3) p(\sigma_3, \sigma_4) \ldots r(\sigma_n, \sigma_1), \tag{16}$$

where

$$p(a, b) = f(a, b) \tilde{f}(a, b), \qquad r(a, b) = g(a, b) \tilde{g}(a, b), \tag{17}$$

$$S = \sum_{\sigma_1, \ldots, \sigma_n} p(\sigma_1, \sigma_2) r(\sigma_2, \sigma_3) \ldots r(\sigma_n, \sigma_1). \tag{18}$$

However, S is just the partition function of a one-dimensional chain of spins with alternating nearest-neighbour interactions. Further, from (16) we see that $\langle L \rangle$ is obtained by averaging L over this one-dimensional system.

It follows that if (10) and (13) are satisfied (with non-negative f, g, $\tilde{f}$, $\tilde{g}$), then intra-row correlations of the IRF model are the same as those of the one-dimensional model with partition function S. The latter model must be disordered (except possibly at zero temperature), so intra-row correlations must decay exponentially. In this sense the IRF model must then be disordered.

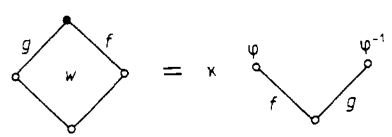

Figure 3. Pictorial representation of equation (10): the function w is associated with the face; f, g with edges; and φ, φ^{-1} with sites. The full circle denotes a spin over whose values the LHS is to be summed; open circles denote spins free to take any value.

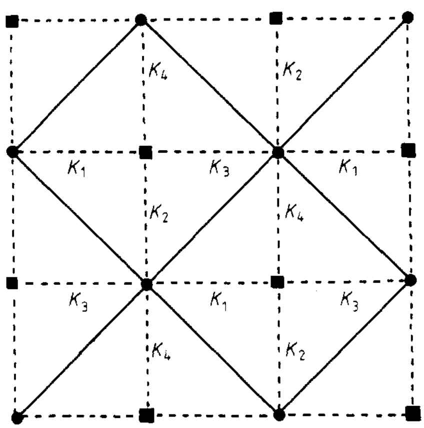

Figure 4. The square lattice $\mathscr{L}$ (broken lines, circles and squares) of the checkerboard Potts model, and the lattice $\mathscr{L}'$ (full lines and circles) of the corresponding IRF model.

Now consider the checkerboard Potts model (Jaekel and Maillard 1984) on the square lattice $\mathscr{L}$ of broken lines shown in figure 4. Each spin σ_i takes the values $1, 2, \ldots, q$ and adjacent spins σ_i, σ_j interact with energy $-k_B T K_r \delta(\sigma_i, \sigma_j)$, where k_B is Boltzmann's constant, T the temperature and K_r is a dimensionless interaction coefficient associated with the edge (i, j). There are four coefficients $K_1, \ldots, K_4$, as indicated in figure 4. The partition function is

$$Z = \sum_{\{\sigma\}} \prod_{(i,j)} \exp[K_r \delta(\sigma_i, \sigma_j)], \tag{19}$$

where the product is over all edges (i, j) of $\mathscr{L}$, K_r being the associated coefficient.

First sum over all spins on sites denoted by small squares in figure 4. We obtain a homogeneous IRF model on the remaining lattice $\mathscr{L}'$ made up of the full circles and lines in figure 4, with weight function

$$w(a, b, c, d) = \sum_s \exp(K_1\delta_{as} + K_2\delta_{bs} + K_3\delta_{cs} + K_4\delta_{ds}) \tag{20}$$

(written a, b, c, d for the four spins round a face of $\mathscr{L}'$, and s for the summed centre spin). The partition function Z is now again given by (1).

We now look for solutions of (10) of the form

$$f(a, b) = F^{\delta(a,b)}, \qquad g(a, b) = G^{\delta(a,b)}, \qquad \varphi(a) = 1. \tag{21}$$

(Thus $f(a, b) = F$ if $a = b$; while $f(a, b) = 1$ if $a \neq b$.)

Thus we have three unknowns at our disposal: κ, F and G. By considering the cases when σ_1, σ_2, σ_3 are all equal, two of them are equal, or all are different, we obtain five apparently distinct equations from (10). This suggests that we shall have to put two constraints on w, i.e. on the interaction coefficients $K_1, \ldots, K_4$.

In fact Jaekel and Maillard (1984) have conjectured that the model has a disorder point provided only one constraint (their equation (6)) is satisfied, and give the corresponding value of κ. We can use these conjectures to find that they predict $F = 1$, i.e.

$$f(a, b) = 1. \tag{22}$$

Thus we can ignore the function f in (10) and in figure 3. It is now quite easy to verify Jaekel and Maillard's conjecture. Set

$$w_i = \exp K_i, \qquad i = 1, \ldots, 4, \tag{23}$$

and write a, b, c, d for σ_1, σ_2, σ_3, σ_2'. Using (20), the equation (10) becomes

$$\sum_s w_2^{\delta(b,s)} w_3^{\delta(c,s)} w_1^{\delta(a,s)} \sum_d G^{\delta(a,d)} w_4^{\delta(d,s)} = \kappa G^{\delta(b,c)}. \tag{24}$$

Consider first the d-sum on the LHS. This depends only on a and s, and can readily be performed to give

$$(q-2+G+w_4)[(q-1+Gw_4)/(q-2+G+w_4)]^{\delta(a,s)}. \tag{25}$$

From (24), we want the LHS to be independent of a; this will be so if (25) is cancelled (as regards its a, s dependence) by the term $w_1^{\delta(a,s)}$, i.e. if

$$w_1(q-1+Gw_4)/(q-2+G+w_4) = 1. \tag{26}$$

The LHS then depends only on b and c, and the s-summation can readily be performed. We find that the LHS of (10) is

$$(q-2+w_2+w_3)(q-2+G+w_4)[(q-1+w_2w_3)/(q-2+w_2+w_3)]^{\delta(b,c)}. \tag{27}$$

Comparing this with the RHS, we see that (24), and hence (28), is satisfied if

$$\tilde{G} = (q-1+w_2w_3)/(q-2+w_2+w_3), \tag{28}$$

$$\kappa = (q-2+w_2+w_3)(q-2+G+w_4). \tag{29}$$

Substituting (28) into (29) and (26), we obtain

$$\kappa = q^2 - 3q + 3 + (q-2)(w_2+w_3+w_4) + w_2w_3 + w_3w_4 + w_4w_2, \tag{30}$$

$$w_1 = \kappa/[(q-1)(q-2+w_2+w_3+w_4) + w_2w_3w_4], \tag{31}$$

or equivalently,

$$\kappa = (q-1+w_1^{-1})^{-1} \prod_{i=1}^{3} (q-1+w_i), \tag{32}$$

$$\frac{w_1 - 1}{1+(q-1)w_1} = \prod_{i=1}^{3} \frac{1-w_i}{q-1+w_i}. \tag{33}$$

Thus if $w_1, \ldots, w_4$ are related by (33) (and $w_1, \ldots, w_4$, G are real and non-negative), then κ is given by (32).

Noting that Jaekel and Maillard's a, b, c, d are our w_2, w_3, w_4, w_1, and that their Z is our $\kappa^{1/2}$ (since the Potts model lattice $\mathscr{L}$ has twice as many sites as the IRF model lattice $\mathscr{L}'$), we observe that (32) and (33) are the same as Jaekel and Maillard's equations (5) and (6). Thus we have proved their conjecture.

Although w_4 enters (26)-(29) in quite a different way from w_2 and w_3, the final results (30)-(33) are symmetric in these three parameters. This is consistent with Jaekel and Maillard's intriguing suggestion that perhaps Z is in general a symmetric function of w_1, w_2, w_3, w_4.

It also means that we can satisfy (13) simultaneously with (10); (13) can be obtained from (10) by interchanging K_2 with K_4, and replacing φ, f, g by $\tilde{\varphi}$, $\tilde{g}$, $\tilde{f}$. This leaves (29)-(33) unchanged, while (22), (21) and (28) become

$$\tilde{g}(a, b) = 1, \qquad \tilde{f}(a, b) = \tilde{F}^{\delta(a,b)} \tag{34, 35}$$

$$\tilde{F} = (q-1+w_3 w_4)/(q-2+w_3+w_4). \tag{36}$$

Thus $p(a, b)$ and $r(a, b)$ in (17) are

$$p(a, b) = \tilde{F}^{\delta(a,b)}, \qquad r(a, b) = G^{\delta(a,b)}. \tag{37}$$

Provided the disorder point condition (33) is satisfied, it follows that intra-row correlations of the Potts model (involving only the spins $\sigma_1, \sigma_2, \ldots, \sigma_n$ in figure 2) are those of the one-dimensional model with partition function (18). In particular, the two-spin correlation function is

$$C_{ij} = [q\langle\delta(\sigma_i, \sigma_j)\rangle - 1]/(q-1). \tag{38}$$

Define, for all integers k,

$$\begin{aligned} P &= (\tilde{F}-1)/(\tilde{F}+q-1) \\ &= (w_3-1)(w_4-1)/[(w_3+q-1)(w_4+q-1)], \end{aligned} \tag{39}$$

$$\begin{aligned} R &= (G-1)/(G+q-1) \\ &= (w_2-1)(w_3-1)/[(w_2+q-1)(w_3+q-1)], \end{aligned}$$

$$D_{2k} = (PR)^k, \qquad D_{2k+1} = (PR)^k R. \tag{40}$$

Then from (16), (18) and (37) we find that (in the limit of n large)

$$C_{ij} = D_j / D_i. \tag{41}$$

In particular, if i and j have the same parity

$$C_{ij} = (PR)^{(j-i)/2}. \tag{42}$$

Since P and R are numerically less than one, we see that the correlation C_{ij} decays exponentially to zero as the spin separation $j-i$ increases. This means that the correlation length is finite and there is no long-range order: the system is disordered (at least in the horizontal direction).

Summary. The general IRF model, with weight function w, is at a disorder point if there exist κ, $\varphi(a)$ and non-negative functions $f(a, b)$, $g(a, b)$ such that (10) is satisfied. The partition function Z is then given by (11). If (13) can also be satisfied (which often is the case), then the intra-row correlations (involving only the spins $\sigma_1, \sigma_2, \ldots, \sigma_n$ in figure 2) are those of the one-dimensional nearest-neighbour model with partition function (18). Such correlations must therefore decay exponentially with spin-separation, and there can be no long-range order in the horizontal direction.

In this letter these general considerations have been applied to the checkerboard Potts model. We have verified in (32) and (33) the disorder-point conjecture of Jaekel and Maillard (1984), and have obtained the simple explicit formula (41) for the two-spin intra-row correlation function.

References

Baxter R J 1980 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
Dhar D 1982 *Phys. Rev. Lett.* **49** 959-62
Enting I G 1977 *J. Phys. A: Math. Gen.* **10** 1023, 1737
—— 1978 *J. Phys. A: Math. Gen.* **11** 555, 2001
Frobenius F G 1908 reprinted in § 79 of *Ferdinand Georg Frobenius*; *Gesammelte Abhandlungen* (Berlin: Springer) (1968)
Jaekel M T and Maillard J M 1984 *Inversion Relation and Disorder Solutions on Potts Models* (preprint)
Peschel I and Rys F 1982 *Phys. Lett.* **91A** 187-9
Rujan P 1982 *J. Stat. Phys.* **29** 247-62
Stephenson J 1970 *Phys. Rev.* B **1** 4405-9
Verhagen A M W 1976 *J. Stat. Phys.* **15** 219
Welberry T R and Galbraith R 1973 *J. Appl. Cryst.* **6** 87
Welberry T R and Miller G H 1978 *Acta Crystallogr.* A **34** 120